On Three Remarkable Affine Connexions in Almost-Hermitian Spaces

by

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§ 1. Almost-Hermitian, almost-Kählerian, pseudo-Hermitian and pseudo-Kählerian spaces.

Let X_{2n} be a 2n-dimensional differentiable manifold of class C^3 admitting an almost complex structure 1) defined by the tensor field $F_{i}^{\cdot h}$ of class C^2:

$$F_{j}^{\cdot h} F_{i}^{\cdot j} = - A_{i}^{h} , \tag{1.1}$$

where the Latin indices h, i, j, k, l, m run over the range 1,2,...., n,n+1,....,2n and A_{i}^{h} denotes the unit tensor. 2)

Since the eigenvalues of the tensor $F_{i}^{\cdot h}$ satisfying (1.1) are +i and -i, the introduction of a tensor $F_{i}^{\cdot h}$ satisfying (1.1) is equivalent to the introduction of two mutually linearly independent linear fields X_{2n}^{n}'s complex conjugate to each other and of complex dimensions n.

A. Lichnerowicz [11,12] showed that it is always possible to give an almost-Hermitian structure g_{ih} to an almost complex manifold X_{2n}:

$$F_{i}^{\cdot l} F_{h}^{\cdot k} g_{lk} = g_{ih} . \tag{1.2}$$

If we put

$$F_{ih} = F_{i}^{\cdot k} g_{kh}, \tag{1.3}$$

then it is easily seen that F_{ih} is anti-symmetric in its lower indices.

We call almost-Hermitian space a manifold which admits an almost complex structure $F_{i}^{\cdot h}$ and an almost-Hermitian structure g_{ih}, that is, a positive definite metric $ds^2 = g_{ih}\, d\xi^{i}\, d\xi^{h}$ satisfying (1.1) and (1.2).

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1) The notion of an almost complex structure was introduced by C. Ehresmann. See for instance, C. Ehresmann [4,5]. The number between brackets refer to the Bibliography at the end of the paper.

2) As to the notations, we follow J.A. Schouten [15].

The anti-symmetry of the tensor F_{ih} and equation (1.2) show that the operation $v^h \to F_i^{\cdot h} v^i$ changes a vector v^h into a vector orthogonal to it and does not change its length.

We now put

$$(1.4) \qquad F_{jih} = 3\, \partial_{[j} F_{ih]}$$

then F_{jih} is a covariant tensor anti-symmetric in all its indices.

Denoting by $\mathring{\nabla}_j$ the operation of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ih}, we can write (1.4) also in the form

$$(1.5) \qquad F_{jih} = 3\, \mathring{\nabla}_{[j} F_{ih]} .$$

In an almost-Hermitian space, when the tensor F_{jih} vanishes identically, the space is called an almost-Kählerian space.

It is well-known that

<u>Theorem 1.1. In order that an almost complex structure $F_i^{\cdot h}$ of class C^ω in a space of class C^ω be induced by a complex structure of the space, it is necessary and sufficient that the tensor $F_i^{\cdot h}$ satisfies</u>

$$(1.6) \qquad N_{ji}^{\cdot\cdot h} \overset{\text{def}}{=} 2\, F_{[j}^{\cdot l}\left(\partial_{|l|} F_{i]}^{\cdot h} - \partial_{i]} F_l^{\cdot h}\right) = 0.$$

The theorem essentially equivalent to this was stated by de Rham, Ehresmann, Libermann [10], Eckmann, Frölicher [3], Calabi, Spencer [1], Guggenheimer [6,7] and Hodge [8]. The tensor $N_{ji}^{\cdot\cdot h}$ was introduced by Nijenhuis [13] 3) and the equation of the form (1.6) was given by Eckmann and Frölicher [3].

Equqtion (1.6) can be written also in the form

$$(1.7) \qquad N_{ji}^{\cdot\cdot h} = 2\, F_{[j}^{\cdot l}\left(\mathring{\nabla}_{|l|} F_{i]}^{\cdot h} - \mathring{\nabla}_{i]} F_l^{\cdot h}\right)$$

or in covariant form

$$(1.8) \qquad N_{jih} \overset{\text{def}}{=} N_{ji}^{\cdot\cdot k} g_{kh} = 2\, F_{[j}^{\cdot l}\left(\mathring{\nabla}_{|l|} F_{i]h} - \mathring{\nabla}_{i]} F_{lh}\right) .$$

It seems quite reasonable to expect that Theorem 1.1 may be true for a tensor $F_i^{\cdot h}$ of class C^{r-1} in a space of class C^r where $r \neq \omega$, but we do not yet have the proof of this fact.

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3) See also J.A. Schouten 14 .

In an almost-Hermitian (almost-Kählerian) space of class C^r, when the Nijenhuis tensor $N_{ji}^{\cdot\cdot h}$ vanishes identically, the space is called a pseudo-Hermitian (pseudo-Kählerian) space. Following Theorem 1.1, a pseudo-Hermitian (pseudo-Kählerian) space of class C is Hermitian (Kählerian).

The relations between these spaces may be seen in the following diagram:

Almost complex structure (X_{2n} of class C^2 with $F_j^{\cdot h} F_i^{\cdot j} = - A_i^h$)	⟶ g_{ih} with $F_i^{\cdot l}F_h^{\cdot k}g_{lk}=g_{ih}$	Almost Hermitian structure	⟶ $\partial_{[j}F_{ih]}=0$	Almost Kählerian structure
↓ $N_{ji}^{\cdot\cdot h}=0$		↓ $N_{ji}^{\cdot\cdot h}=0$		↓ $N_{ji}^{\cdot\cdot h}=0$
Pseudo-complex structure	⟶ g_{ih} with $F_i^{\cdot l}F_h^{\cdot k}g_{lk}=g_{ih}$	Pseudo-Hermitian structure	⟶ $\partial_{[j}F_{ih]}=0$	Almost Kählerian structure
↓ $r=\omega$		↓ $r=\omega$		↓ $r=\omega$
Complex structure	⟶ g_{ih} with $F_i^{\cdot l}F_h^{\cdot k}g_{lk}=g_{ih}$	Hermitian structure	⟶ $\partial_{[j}F_{ih]}=0$	Kählerian structure

In an almost-Hermitian space, it is easily seen from (1.5) and (1.8) that if $\overset{\circ}{\nabla}_j F_{ih}$ vanishes, then the tensors F_{jih} and N_{jih} vanish also.

Conversely, since we have from (1.5) and (1.8)

$$N_{jih} = 2\,(F_j^{\cdot l}\, \overset{\circ}{\nabla}_h F_{il} - F_{[j}^{\cdot l}\, F_{i]hl}), \tag{1.9}$$

we can easily see that if two tensors F_{jih} and N_{jih} vanish, then $\overset{\circ}{\nabla}_j F_{ih}$ vanishes also. Thus we have [17].

<u>Theorem 1.2. A necessary and sufficient condition that an almost-Hermitian space be pseudo-Kählerian is that $\overset{\circ}{\nabla}_j F_{ih} = 0$.</u>

§ 2. Affine connexions leaving invariant the tensor F_{ih}.

Let us now introduce an affine connexion Γ^h_{ji} in our almost-Hermitian space and put

(2.1) $$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + T_{ji}^{\cdot\cdot h},$$

then the torsion tensor is given by

(2.2) $$S_{ji}^{\cdot\cdot h} = \Gamma^h_{[ji]} = T_{[ji]}^{\cdot\cdot\ h}.$$

Denoting by ∇_j the operation of covariant differentiation with respect to Γ^h_{ji}, in order to have $\nabla_j g_{ih} = 0$, it is necessary and sufficient that we have

(2.3) $$T_{j(ih)} = 0,$$

which is equivalent to

(2.4) $$T_{jih} = S_{jih} + S_{hij} + S_{hji},$$

where

(2.5) $$T_{jih} \overset{def}{=} T_{ji}^{\cdot\cdot k} g_{kh}, \quad S_{jih} \overset{def}{=} S_{ji}^{\cdot\cdot k} g_{kh}.$$

On the other hand, inórder to have $\nabla_j F_{ih} = 0$, it is necessary and sufficient that we have

(2.6) $$\overset{\circ}{\nabla}_j F_{ih} - T_{ji}^{\cdot\cdot l} F_{lh} - T_{jh}^{\cdot\cdot l} F_{il} = 0$$

or what amounts to the same

(2.7) $$\overset{\circ}{\nabla}_j F_{ih} + T_{jil} F_h^{\cdot l} - T_{jhl} F_i^{\cdot l} = 0.$$

We cannot solve equation (2.7) without putting further conditions on T_{jih}. Putting some further conditions on T_{jih} or rather on

(2.8) $$T_{jih} \overset{def}{=} T_{jil} F_h^{\cdot l},$$

we shall try to solve equation (2.7) with respect to T_{jih}. From now on, we assume only $\nabla_j F_{ih} = 0$ and not $\nabla_j g_{ih} = 0$.

§ 3. The first connexion.

We first assume that

(3.1) $$\overset{\prime}{T}_{j(ih)} = 0,$$

then we have from (2.7)

(3.2) $$\overset{\prime}{T}_{jih} = -\tfrac{1}{2}\overset{\circ}{\nabla}_j F_{ih}$$

or

(3.3) $$\overset{\prime}{T}_{jih} = \tfrac{1}{2}(\overset{\circ}{\nabla}_j F_{il})F_h^{\cdot l}$$

and consequently

(3.4) $$\overset{\prime}{T}_{ji}^{\cdot\cdot h} = -\tfrac{1}{2}(\overset{\circ}{\nabla}_j F_{il})F^{lh}.$$

From (3.4), we get

(3.5) $$\overset{\prime}{S}_{ji}^{\cdot\cdot h} = \overset{\prime}{T}_{ji}^{\cdot\cdot h} = -\tfrac{1}{2}(\overset{\circ}{\nabla}_j F_{il})F^{lh}.$$

Since $F_{il}F_h^{\cdot l} = g_{ih}$ and consequently

$$(\overset{\circ}{\nabla}_j F_{il})F_h^{\cdot l} + F_i^{\cdot l}(\overset{\circ}{\nabla}_j F_{hl}) = 0,$$

equation (3.3) shows that $T_{j(ih)} = 0$ and consequently the connexion is metric and we have

(3.6) $$\overset{\prime}{\nabla}_j g_{ih} = 0 \quad , \quad \overset{\prime}{\nabla}_j F_{ih} = 0.$$

From (1.9) and (3.3), we find

(3.7) $$\overset{\prime}{T}_{jih} = \tfrac{1}{4}N_{hij} - \tfrac{1}{2}F_{[h}^{\cdot l}F_{i]gl}.$$

Thus we have, for an almost-Kählerian space,

(3.8) $$\overset{\prime}{T}_{jih} = \tfrac{1}{4}N_{hij},$$

for a pseudo-Hermitian space,

(3.9) $$\overset{\prime}{T}_{jih} = -\tfrac{1}{2}F_{[h}^{\cdot l}F_{i]jl},$$

and for a pseudo-Kählerian space,

(3.10) $$\overset{\prime}{T}_{jih} = 0.$$

In the last case the connexion Γ^{h}_{ji} becomes Riemannian.

In a Hermitian space tensors g_{ih} and $F_i^{\cdot h}$ have the components

$$(3.11)\qquad g_{ih} = \begin{matrix} 0 & g_{\ x} \\ g_{\ x} & 0 \end{matrix}\ , \qquad F_i^{\cdot h} = \begin{matrix} +i\ ^{x} & 0 \\ 0 & -i\ ^{x} \end{matrix}$$

respectively and consequently the tensors F_{ih} and F^{ih} def gil $F_i^{\cdot h}$ the components

$$(3.12)\qquad F_{ih} = \begin{matrix} 0 & -ig_{\ x} \\ +ig_{\ x} & 0 \end{matrix}\ , \quad F^{ih} = \begin{matrix} 0 & +ig^{\ x} \\ -ig\ x & 0 \end{matrix}$$

respectively with respect to a complex coordinate system (z^x, z^x z^x), where the Greek indices x, , , , , run over the range 1,2,...,n and x, , , , , the range n+1, n+2,...,2n.

Thus for the Christoffel symbols $\{^{h}_{ji}\}$ we have [4)]

$$(3.13)\qquad {}^{x} = g^{\ x}(\ g\ )\ , \quad {}^{x} = g^{\ x}\ g\ , \quad {}^{x} = 0,$$

$$x = g\ ^{x}\ g$$

and four similar expressions for x , x , x and x .
We shall denote in the following by the sign "conj." the fact that we have similar formulas as complex conjugates of the formulas already written.

For the tensor F_{jih} and ${}_{j}F_{ih}$ we have respectively

$$F_{\ x} = 0, \qquad \text{conj.}$$

$$(3.14)\qquad F_{\ x} = +2i\ \ g_{\ \ x} \qquad \text{conj.}$$

$$F_{\ x} = +2\,i\ {}_{x}g \qquad \text{conj.}$$

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4) See for example J.A. Schouten 15 , P. 396.

and

$$(3.15)\quad \begin{aligned} F^{\cdot x} &= 0, && \text{conj.}\\ F^{\cdot x} &= -2i\ \ {}^{x} && \text{conj.}\\ F^{\cdot x} &= 0, && \text{conj.}\\ F^{\cdot x} &= 0. && \text{conj.}\end{aligned}$$

Thus for the tensor (3.4), we have

$$(3.16)\qquad T^{\cdot\cdot x} = 0,\ T^{\cdot\cdot x} = -\ \ {}^{x}\ ,\ T^{\cdot\cdot x} = 0,\ T^{\cdot\cdot x} = 0 \qquad \text{conj.}$$

and consequently, for the components of the affine connexion

$$(3.17)\qquad {}^{h}_{ji} = {}^{h}_{ji} + T_{ji}^{\cdot\cdot h},$$

we have

$$(3.18)\qquad {}^{x} = {}^{x}\ ,\quad {}^{x} = {}^{x} \qquad \text{conj.}$$

the other 's being zero.

This is the affine connexion introduced by A. Lichnerowicz 12 in an almost-Hermitian space. In a Hermitian space, this connexion seems to be a natural one because this connexion is obtained from Riemannian connection ${}^{h}_{ji} = {}^{h}_{ji}$ requiring ${}_{j}F_{i}^{\cdot h} = 0$, that is, requiring ${}_{jx} = {}_{jx} = 0$, the other 's keeping the same values.

In a classification by Schouten (15 p. 396, formula (3.6)), this corresponds to the case

$$(3.19)\qquad S^{\cdot\cdot x} = 0,\ S^{\cdot\cdot x} = -\tfrac{1}{2}\ g\ {}^{x}\ ,\ S^{\cdot\cdot x} = 0, \qquad \text{conj.}$$

§ 4. The second connexion.

We next assume that

$$(4.1)\qquad T\,(j\ i\ h) = 0,$$

then we have from (2.7)

$$(4.2)\qquad T_{jih} = -\tfrac{1}{2}\,(\ {}_{j}F_{ih} + {}_{i}F_{jh} - {}_{h}F_{ij})$$

or

(4.3) $$T_{jih} = + \frac{1}{2} ({}_j F_{il} + {}_i F_{jl} - {}_l F_{ij}) F_h^{\cdot l}$$

and consequently

(4.4) $$T_{ji}^{\cdot\cdot h} = - \frac{1}{2} ({}_j F_{il} + {}_i F_{jl} - {}_l F_{ij}) F^{lh} .$$

From (4.4), we get

(4.5) $$S_{ji}^{\cdot\cdot h} = T_{\ ji}^{\cdot\cdot h} = - \frac{1}{2} ({}_l F_{ji}) F^{lh} .$$

From (4.3), we find

(4.6) $$T_{jih} = F_h^{\cdot l} \ {}_i F_{jl} - \frac{1}{2} F_h^{\cdot l} F_{ijl} .$$

Thus combining (1.9) and (4.6), we find

(4.7) $$T_{jih} = \frac{1}{2} N_{hji} - \frac{1}{2} F_j^{\cdot l} F_{ihl} .$$

Thus we have, for an almost-Kählerian space,

(4.8) $$T_{jih} = \frac{1}{2} N_{hji} .$$

and for a pseudo-Hermitian space

(4.9) $$T_{jih} = - \frac{1}{2} F_j^{\cdot l} F_{ihl} .$$

In the case of pseudo-Hermitian space, the tensor T_{jih} satisfies (2.3) and consequently the connexion is metric and we have

(4.10) $${}_j g_{ih} = 0, \qquad {}_j F_{ih} = 0.$$

Finally we have, for a pseudo-Kählerian space,

(4.11) $$T_{jih} = 0$$

and the connexion becomes Riemannian.

In a Hermitian space, in addition to formulas (3.11)-(3.15), we have following expressions for ${}_j F_{ih}$:

(4.12) $$F_{\ x} = 0, \qquad F_{\ x} = 0, \qquad F_{\ x} = 0,$$

$$F_{\ x} = - 2 i \qquad\qquad \text{conj.}$$

Thus, for the tensor T_{ji}, we have

$$(4.13)\quad \begin{aligned} &T^{\cdot\cdot x} = g \quad x \qquad g \qquad , && \text{conj.} \\ &T^{\cdot\cdot x} = - g \quad x \qquad g \qquad , && \text{conj.} \\ &T^{\cdot\cdot x} = 0, && \text{conj.} \\ &T^{\cdot\cdot x} = - g \quad x \qquad g \qquad , && \text{conj.} \end{aligned}$$

and consequently for the connexion

$$(4.14)\qquad {}^{h}_{ji} = {}^{h}_{ji} + T_{ji}^{\cdot\cdot h}$$

we have

$$(4.15)\qquad {}^{x} = g \quad x \quad g \quad , \quad {}^{x} = 0, \quad {}^{x} = 0, \quad {}^{x} = 0, \ \text{conj.}$$

This is the affine connexion introduced by J.A. Schouten and D. van Dantzig 16 and used recently by S.S. Chern 2 and P. Libermann 10 [5].

In a classification by Schouten, (15 , p. 396, formula (3.6)), this corresponds to the case

$$(4.16)\qquad S^{\cdot\cdot x} = g \qquad g \quad , \ S \ {}^{x} = 0, \ S^{\cdot\cdot x} = 0 \qquad \text{conj.}$$

§ 5. The third connexion.

We finally assume that

$$(5.1)\qquad T_{(ji)h} = 0,$$

then we have from (2.7)

$$(5.2)\qquad T_{jih} = - \tfrac{1}{2} (\ {}_{j} F_{ih} - {}_{i} F_{jh} + {}_{h} F_{ij})$$

or

$$(5.3)\qquad T_{jih} = + \tfrac{1}{2} (\ {}_{j} F_{il} - {}_{i} F_{jl} - {}_{l} F_{ij}) F_{h}^{\cdot l}$$

and consequently

$$(5.4)\qquad T_{ji}^{\cdot\cdot h} = - \tfrac{1}{2} (\ {}_{j} F_{il} - {}_{i} F_{jl} + {}_{l} F_{ij}) F^{lh} .$$

5) See also G. Legrand 9 .

From (5.4) we get

(5.5) $$S_{ji}^{\cdot\cdot h} = T\ {}_{ji}^{\cdot\cdot h} = T_{ji}^{\cdot\cdot h} .$$

From (5.3) we find

(5.6) $$N_{jih} = 2(\ {}_j F_{i\,l}) F_h^{\cdot l} + F_j^{\cdot l}\ {}_l F_{ih} + F_i^{\cdot l}\ {}_l F_{hj} .$$

Thus combining (5.3) and (5.6), we obtain

(5.7) $$T_{jih} = \tfrac{1}{2} N_{jih} + \tfrac{1}{2} (F_j^{\cdot l}\ {}_l F_{ih} + F_i^{\cdot l}\ {}_l F_{hj} + F_h^{\cdot l}\ {}_l F_{ji}) .$$

On the other hand, we have from (1.5)

(5.8) $$F_j^{\cdot m} F_i^{\cdot l} F_h^{\cdot k} F_{mlk} = - 3 F\ {}_j^{l}\ {}_l F_{ih} .$$

Substituting this into (5.7), we find

(5.9) $$T_{jih} = \tfrac{1}{2} N_{jih} + \tfrac{1}{2} F_j^{\cdot m} F_i^{\cdot l} F_h^{\cdot k} F_{mlk} .$$

Thus we have, for an almost-Kählerian space,

(5.10) $$T_{jih} = \tfrac{1}{2} N_{jih}$$

and, for a pseudo-Hermitian space,

(5.11) $$T_{jih} = \tfrac{1}{2} F_j^{\cdot m} F_i^{\cdot l} F_h^{\cdot k} F_{mlk} .$$

In the case of pseudo-Hermitian space, the tensor T_{jih} is anti-symmetric in all its indices and consequently satisfies (2.3) and the connexion is metric.

Thus we have

(5.12) $$ {}_j g_{ih} = 0, \qquad {}_j F_{ih} = 0.$$

Finally we have, for a pseudo-Kählerian space,

(5.13) $$T_{jih} = 0,$$

and the connexion becomes Riemannian.

In a Hermitian space, we have, for the tensor $T_{ji}^{\cdot\cdot h}$,

$$T^{\cdot\cdot x} = - g^{x} \quad g \qquad \text{conj.}$$

$$(5.14) \qquad T^{\cdot\cdot x} = - g^{x} \qquad \text{conj.}$$

$$T^{\cdot\cdot x} = 0, \qquad \text{conj.}$$

$$T^{\cdot\cdot x} = + g^{x} \qquad \text{conj.}$$

and consequently, for the components of the affine connection

$$(5.15) \qquad {}^{h}_{ji} = {}^{h}_{ji} + T_{ji}^{\cdot\cdot h} ,$$

we have

$$(5.16) \qquad {}^{x} = g \quad g \;, \quad {}^{x} = 0, \quad {}^{x} = 0, \quad {}^{x} = 2\, g \quad g$$

conj.

In a classification by J.A. Schouten (15 p. 396, formula (3.6)), this corresponds to the case

$$(5.17) \qquad S^{\cdot\cdot x} = - g \quad g \;, \quad S^{\cdot\cdot x} = - g \quad g \;,$$

$$S^{\cdot\cdot x} = 0, \qquad \text{conj.}$$

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